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Cardinal characteristics at κ in a small $\mathfrak{u}(\kappa)$ model

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Abstract

We provide a model where $\mathfrak{u}(\kappa) < 2^\kappa$ for a supercompact cardinal κ . [10] provides a sketch of how to obtain such a model by modifying the construction in [6]. We provide here a complete proof using a different modification of [6] and further study the values of other natural generalizations of classical cardinal characteristics in our model. For this purpose we generalize some standard facts that hold in the countable case as well as some classical forcing notions and their properties.

Keywords: generalized cardinal characteristics, forcing, supercompact cardinals

2000 MSC: 03E17, 03E35, 03E55

1. Introduction

Cardinal characteristics on the Baire space ω^ω have been widely studied and understood. Since 1995 with the Cummings-Shelah paper [5], the study of the generalization of these cardinal notions to the context of uncountable cardinals and their interactions has been developing. By now, there is a wide literature on this topic. Some key references (at least for the purposes of this paper) are [2], [5] and [14].

In [6] Džamonja and Shelah construct a model with a universal graph at the successor of a strong limit singular cardinal of countable cofinality. A variant of this model, as pointed out by Garti and Shelah in [10], witnesses the consistency of $\mathfrak{u}(\kappa) = \kappa^+ < 2^\kappa$ (Here $\mathfrak{u}(\kappa) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is an base for a uniform ultrafilter on } \kappa\}$). See also [4].

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Here we present a modification of the forcing construction introduced by Džamonja and Shelah in [6], which allows us to prove that if κ is a supercompact cardinal and $\kappa < \kappa^*$ with κ^* regular, then there is a generic extension of the universe in which cardinals have not been changed and $\mathfrak{u}(\kappa) = \kappa^*$. As in [6], we use a long iteration of forcings each with an initial choice of ultrafilter, and truncate the iteration at an appropriate point of the desired cofinality (for us, κ^*). Specifically, we truncate at a limit of stages at which the chosen ultrafilter is the canonical one given by an elementary embedding j : namely, those X such that $\kappa \in j(X)$. The proof in [6] that such a stage exists is by a careful, bottom-up recursion. We take a more top-down approach (for which see also [8]) that freely refers to the canonical ultrafilter in the eventual full generic extension, of which the desired ultrafilters are just restrictions. We believe this perspective makes for a more streamlined proof. As in [6], we may in addition ensure that each of the restricted ultrafilters contains a Mathias generic for its smaller restrictions, yielding an ultrafilter generated by these κ^* -many Mathias generics and so obtaining $\mathfrak{u}(\kappa) = \kappa^*$.

Moreover our construction allows us to decide the values of many of the higher analogues of the known classical cardinal characteristics of the continuum, as we can interleave arbitrary κ -directed closed posets cofinally in the iteration. The detailed construction of our model is presented in Section 3, while our applications appear in Section 4.

Thus our main result states the following:

Theorem 1. *Suppose κ is a supercompact cardinal, κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$ and Γ satisfies $\Gamma^\kappa = \Gamma$. Then there is forcing extension in which cardinals have not been changed satisfying:*

$$\begin{aligned} \kappa^* = \mathfrak{u}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \text{cov}(\mathcal{M}_\kappa) \\ = \text{add}(\mathcal{M}_\kappa) = \text{non}(\mathcal{M}_\kappa) = \text{cof}(\mathcal{M}_\kappa) \text{ and } 2^\kappa = \Gamma. \end{aligned}$$

If in addition $(\Gamma)^{<\kappa^} \leq \Gamma$ then we can also provide that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{h}_\mathcal{W}(\kappa) = \kappa^*$ where \mathcal{W} is a κ -complete ultrafilter on κ .*

We also establish some of the natural inequalities between the generalized characteristics which in the countable case are well known.

2. Preliminaries

Let κ be a supercompact cardinal. Recall that this means that for all $\lambda \geq \kappa$ there is an elementary embedding $j : V \rightarrow M$ with critical point κ , $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$.

One of the main properties of supercompact cardinals that will be used throughout the paper is the existence of the well-known Laver preparation, which makes the supercompactness of κ indestructible by subsequent forcing with κ -directed-closed partial orders.

Theorem 2 (Laver, [13]). *If κ is supercompact, then there exists a κ -cc partial ordering S_κ of size κ such that in V^{S_κ} , κ is supercompact and remains supercompact after forcing with any κ -directed closed partial order.*

The main lemma used to obtain this theorem is the statement that for any supercompact cardinal κ there exists a *Laver diamond*. That is, there is a function $h : \kappa \rightarrow V_\kappa$ such that for every set x and every cardinal λ , there is an elementary embedding $j : V \rightarrow M$ with critical point κ , $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $j(h)(\kappa) = x$.

Given such a function, the Laver preparation S_κ is given explicitly as a reverse Easton iteration $(S_\alpha, \dot{R}_\beta : \alpha \leq \kappa, \beta < \kappa)$, defined alongside a sequence of cardinals $(\lambda_\alpha : \alpha < \kappa)$ by induction on $\alpha < \kappa$ as follows.

- If α is a cardinal and $h(\alpha) = (\dot{P}, \lambda)$, where λ is a cardinal, \dot{P} is an S_α name for a α -directed closed forcing, and for all $\beta < \alpha$, $\lambda_\beta < \alpha$, we let $\dot{R}_\alpha := \dot{P}$ and $\lambda_\alpha = \lambda$.
- Otherwise, we let \dot{R}_α be the canonical name for the trivial forcing and $\lambda_\alpha = \sup_{\beta < \alpha} \lambda_\beta$.

One of the main forcing notions we will use is the following:

Definition 3 (Generalized Mathias Forcing). *Let κ be a measurable cardinal, and let \mathcal{F} be a κ -complete filter on κ . The Generalized Mathias Forcing $\mathbb{M}_{\mathcal{F}}^\kappa$ has, as its set of conditions, $\{(s, A) : s \in [\kappa]^{<\kappa} \text{ and } A \in \mathcal{F}\}$, and the ordering given by $(t, B) \leq (s, A)$ if and only if $t \supseteq s$, $B \subseteq A$ and $t \setminus s \subseteq A$. We denote by $\mathbb{1}_{\mathcal{F}}$ the maximum element of $\mathbb{M}_{\mathcal{F}}^\kappa$, that is $\mathbb{1}_{\mathcal{F}} = (\emptyset, \kappa)$.*

In our main forcing iteration construction we work exclusively with generalized Mathias posets $\mathbb{M}_{\mathcal{U}}^\kappa$, where \mathcal{U} is a κ -complete ultrafilter. In our applications however, we will be working with arbitrary κ -complete filters.

Definition 4. *A partial order \mathbb{P} is:*

- κ -centered if there is a partition $\{\mathbb{P}_\alpha \mid \alpha < \kappa\}$ of \mathbb{P} such that for each $\alpha < \kappa$, every pair of conditions $p, q \in \mathbb{P}_\alpha$ has a common extension in \mathbb{P}_α ;
- κ -directed closed if for every directed set $D \subseteq \mathbb{P}$ of size $|D| < \kappa$ there is a condition $p \in \mathbb{P}$ such that $p \leq q$ for all $q \in D$.

3. A small $\mathfrak{u}(\kappa)$ model

Let Γ be such that $\Gamma^\kappa = \Gamma$. We will define an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \Gamma^+, \beta < \Gamma^+ \rangle$ of length Γ^+ recursively as follows:

If α is an even ordinal (abbreviated $\alpha \in \text{EVEN}$), let NUF denote the set of normal ultrafilters on κ in $V^{\mathbb{P}_\alpha}$. Then let \mathbb{Q}_α be the poset with underlying set of conditions $\{\mathbb{1}_{\mathbb{Q}_\alpha}\} \cup \{(\mathcal{U}) \times \mathbb{M}_\mathcal{U}^\kappa : \mathcal{U} \in \text{NUF}\}$ and extension relation stating that $q \leq p$ if and only if either $p = \mathbb{1}_{\mathbb{Q}_\alpha}$, or there is $\mathcal{U} \in \text{NUF}$ such that $p = (\mathcal{U}, p_1)$, $q = (\mathcal{U}, q_1)$ and $q_1 \leq_{\mathbb{M}_\mathcal{U}^\kappa} p_1$. If α is an odd ordinal (abbreviated $\alpha \in \text{ODD}$), let $\dot{\mathbb{Q}}_\alpha$ be a \mathbb{P}_α -name for a κ -centered, κ -directed closed forcing notion of size at most Γ .

We define three different kinds of support for conditions $p \in \mathbb{P}_\alpha$, $\alpha < \Gamma^+$: First we have the *Ultrafilter Support* $\text{USupt}(p)$, that corresponds to the set of ordinals $\beta \in \text{dom}(p) \cap \text{EVEN}$ such that $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \neq \mathbb{1}_{\mathbb{Q}_\beta}$. Then the *Essential Support* $\text{SSupt}(p)$, which consists of all $\beta \in \text{dom}(p) \cap \text{EVEN}$ such that $\neg(p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \{\mathbb{1}_{\mathbb{Q}_\beta}\} \cup \{(\mathcal{U}, \mathbb{1}_\mathcal{U}) : \mathcal{U} \in \text{NUF}\})$ (for the definition of $\mathbb{1}_\mathcal{U}$ see Definition 3). Finally, the *Directed Support* $\text{RSupt}(p)$, consists of all $\beta \in \text{dom}(p) \cap \text{ODD}$ such that $\neg(p \restriction \beta \Vdash p(\beta) = \mathbb{1}_{\dot{\mathbb{Q}}_\beta})$.

We require that the conditions in \mathbb{P}_{Γ^+} have support bounded below Γ^+ and also that given $p \in \mathbb{P}_{\Gamma^+}$ if $\beta \in \text{USupt}(p)$ then for all $\alpha \in \beta \cap \text{EVEN}$, $\alpha \in \text{USupt}(p)$. Finally we demand that both $\text{SSupt}(p)$ and $\text{RSupt}(p)$ have size $< \kappa$ and are contained in $\text{sup}(\text{USupt}(p))$, that is $\text{Supt}(p)$ (the entire support of p) and $\text{USupt}(p)$ have the same supremum.

Now, we want to ensure that our iteration preserves cardinals. Let $\mathbb{P} := \mathbb{P}_{\Gamma^+}$.

Lemma 5. \mathbb{P} is κ -directed closed.

Proof. We know that $\mathbb{M}_\mathcal{U}^\kappa$, as well as all iterands \mathbb{Q}_α for $\alpha \in \text{ODD}$, are κ -directed closed forcings. Take $D = \{p_\alpha : \alpha < \delta < \kappa\}$ a directed set of conditions in \mathbb{P} . We want to define a common extension p for all elements in D . First define $\text{dom}(p) = \bigcup_{\alpha < \delta} \text{dom}(p_\alpha)$. For $j \in \text{dom}(p)$ define $p(j)$ by induction on j . We work in $V^{\mathbb{P}_j}$ and assume that $p \restriction j \in \mathbb{P}_j$.

We have the following cases:

- If j is even and $j \notin \bigcup_{\alpha < \delta} \text{SSupt}(p_\alpha)$, then using directedness we can find a name $\dot{\mathcal{U}}$ for a normal ultrafilter such that for all $\alpha < \delta$, $p \restriction j \Vdash (\dot{\mathcal{U}}, \mathbb{1}_{\dot{\mathcal{U}}}) \leq p_\alpha(j)$. Define $p(j)$ to be the canonical name for $(\mathcal{U}, \mathbb{1}_\mathcal{U})$.
- If j is even and $j \in \text{SSupt}(p_\alpha)$ for some $\alpha < \delta$, then again using directedness it is possible to find a single name $\dot{\mathcal{U}}$ for an ultrafilter

such that for $\alpha < \delta$ with $j \in \text{SSupt}(p_\alpha)$, $p \restriction j \Vdash p_\alpha(j) \in \dot{\mathcal{U}} \times \mathbb{M}_{\dot{\mathcal{U}}}^\kappa$, and $\Vdash_{\mathbb{P}_j} \mathbb{M}_{\dot{\mathcal{U}}}^\kappa$ is κ -directed closed. We can thus find a condition q such that $p \restriction j \Vdash q \leq p_\alpha(j)$ for all $\alpha < \delta$. Define $p(j) = q$.

- If j is odd, use the fact that in the \mathbb{P}_j extension \mathbb{Q}_j is κ -directed closed on the directed set $X_j = \{p_\alpha(j) : \alpha < \delta < \kappa\}$ to find $p(j)$ a condition stronger than all the ones in X_j .

□

For any $p \in \mathbb{P}_\beta$, $\beta < \Gamma^+$ let $\mathbb{P}_\beta \downarrow p$ denote the set $\{q \in \mathbb{P}_\beta : q \leq p\}$.

Lemma 6. *Let $p \in \mathbb{P}_{\Gamma^+}$ and let $i = \sup \text{USupt}(p) = \sup \text{Supt}(p)$. Then $\mathbb{P}_i \downarrow (p \restriction i)$ is κ^+ -cc and has a dense subset of size at most Γ .*

Proof. It is enough to observe that $\mathbb{P}_i \downarrow (p \restriction i)$ is basically a $< \kappa$ -support iteration of κ -centered, κ -directed closed forcings of size at most Γ . Then the proof is a straightforward generalization of Lemma V.4.9 – V.4.10 in [12]. □

Lemma 7. *Let $\{\mathcal{A}_\alpha\}_{\alpha < \Gamma}$ be maximal antichains in \mathbb{P} below $p \in \mathbb{P}$. Let $j^* = \sup \text{Supt}(p)$. Then there is $q \in \mathbb{P}$ such that $q \restriction j^* = p$, $\text{Supt}(q) \setminus \text{Supt}(p) \subseteq \text{USupt}(q) \setminus \text{SSupt}(q)$ and for all $\alpha < \Gamma$, the set $\mathcal{A}_\alpha \cap (\mathbb{P}_{i^*} \downarrow q)$ is a maximal antichain in $\mathbb{P}_{i^*} \downarrow q$ (and hence in $\mathbb{P} \downarrow q$), where $i^* = \sup \text{Supt}(q)$.*

Proof. Let $\bar{\mathbb{P}} := \mathbb{P}_{j^*} \downarrow p$ and let $w \in \bar{\mathbb{P}}$. Then there is a condition r extending both w and an element of \mathcal{A}_0 and we can find p_1 such that $p_1 \restriction j^* = p$, $\text{SSupt}(p_1) \cup \text{RSupt}(p_1) \subseteq j^*$, and $r \in \mathbb{P}_{j_1} \downarrow p_1$, where $j_1 = \sup \text{Supt}(p_1)$. Since $\bar{\mathbb{P}}$ has a dense subset of size at most Γ , in Γ -steps we can find q_0 such that $q_0 \restriction j^* = p$ and every condition in $\bar{\mathbb{P}}$ is compatible with an element of $\mathcal{A}_0 \cap (\mathbb{P}_{j_0^*} \downarrow q_0)$, where $j_0^* = \sup \text{Supt}(q_0)$.

Since we have only Γ many antichains $\{\mathcal{A}_\alpha\}_{\alpha < \Gamma}$, in Γ steps we can obtain the desired condition q . □

Corollary 8. *If $p \Vdash \dot{X} \subseteq \kappa$ for some \mathbb{P} -name \dot{X} , then there are $q \leq p$ and $j < \Gamma^+$ such that \dot{X} can be seen as a $\mathbb{P}_j \downarrow q$ -name.*

Proof. For each $\alpha < \kappa$ fix a maximal antichain \mathcal{A}_α of conditions below p deciding if α belongs to \dot{X} . Then, let q be the condition given by Lemma 7 and take $j := \sup \text{Supt}(q)$. Then $q \leq p$ and \dot{X} can be seen as a $\mathbb{P}_{\sup \text{Supt}(q)} \downarrow q$ -name. □

Corollary 9. *Let $p \Vdash \dot{f}$ is a \mathbb{P} -name for a function from Γ into the ordinals. Then there is a function $g \in V$ and $q \leq p$ such that $q \Vdash \dot{f}(\alpha) \in g(\alpha)$ for $\alpha < \Gamma$ and $|g(\alpha)| \leq \kappa$ for all α . In particular, \mathbb{P} preserves cofinalities and so cardinalities.*

Proof. Let \mathcal{A}_α be a maximal antichain of conditions below p deciding a value for $\dot{f}(\alpha)$. Use Lemma 7 to find $q \leq p$ such that $\mathcal{A}_\alpha \cap \mathbb{P} \downarrow q$ is a maximal antichain in $\mathbb{P} \downarrow q$ for all $\alpha < \Gamma$. Finally define the function $g \in V$ as follows: $g(\alpha) = \{\beta : \exists r \leq q \text{ such that } r \Vdash \dot{f}(\alpha) = \beta\}$. \square

We now present the key lemmas that will allow us to construct the witness for $\mathfrak{u}(\kappa) = \kappa^*$.

Lemma 10. *Let κ be a supercompact cardinal and κ^* be a cardinal satisfying $\kappa < \kappa^* \leq \Gamma$, κ^* regular. Suppose that $p \in \mathbb{P}$ is such that $p \Vdash \dot{\mathcal{U}}$ is a normal ultrafilter on κ .¹ Then for some $\alpha < \Gamma^+$ there is an extension $q \leq p$ such that $q \Vdash (\dot{\mathcal{U}}_\alpha = \dot{\mathcal{U}} \cap V[G_\alpha])$. Moreover this can be done for a set of ordinals $S \subseteq \Gamma^+$ of order type κ^* in such a way that $\forall \alpha \in S (\dot{\mathcal{U}} \cap V_\alpha \in V[G_\alpha])$ and $\dot{\mathcal{U}} \cap V[G_{\sup S}] \in V[G_{\sup S}]$. Here $\dot{\mathcal{U}}_\alpha$ is the canonical name for the ultrafilter generically chosen at stage α .*

Proof. Let $\alpha_0 = \sup \text{USupt}(p)$. Then $\mathbb{P}_{\alpha_0} \downarrow p$ is κ^+ -cc and has a dense subset of size at most Γ . Thus there are just Γ -many $\mathbb{P}_{\alpha_0} \downarrow p$ -names for subsets of κ . Let $\bar{X} = (\dot{X}_i : i < \Gamma)$ be an enumeration of them.

We view each condition in \mathbb{P} as having three main parts. The first part corresponds to the choice of ultrafilters in even coordinates — the “ \mathcal{U} ”s of $r = (\mathcal{U}, r_1)$ for iterand conditions r ; we call this the *Ultrafilter Part*. The next part corresponds to the coordinates where we have in addition non-trivial Mathias conditions (coordinates in SSupt), we call it the *Mathias part*. Finally the odd coordinates, where the forcing chooses conditions in an arbitrary κ -centered, κ -directed closed forcing (coordinates in RSupt), we call the *Directed Part*.

Extend $p_0 = p$ to a condition p_1 deciding whether $\dot{X}_0 \in \dot{\mathcal{U}}$, and let p'_1 be the condition extending p_0 with the same ultrafilter part as p_1 and no other change from p_0 . Then extend p'_1 again to a condition p_2 which also makes a decision about \dot{X}_0 but either its Mathias or directed parts are incompatible with the ones corresponding to p_1 ; and correspondingly extend p'_1 on its ultrafilter part to p'_2 .

¹This will be possible because κ is still supercompact in $V^{\mathbb{P}}$.

Continue extending the ultrafilter part, deciding whether or not $\dot{X}_0 \in \dot{U}$ with an antichain of different Mathias and directed parts until a maximal antichain is reached. This will happen in less than κ^+ -many steps. If the resulting condition is called q_1 and has support $\alpha_1 < \Gamma^+$, then the set of conditions in $\mathbb{P}_{\alpha_1} \downarrow q_1$ which decide whether or not \dot{X}_0 belongs to \dot{U} is predense in $\mathbb{P}_{\alpha_1} \downarrow q_1$.

Repeat this process Γ -many times for each element in \bar{X} until reaching a condition q_2 with the same property for all such names. Then do it for all $\mathbb{P}_{\alpha_1} \downarrow q_2$ names for subsets of κ and so on. Let q be the condition obtained once this overall process closes off with a fixed point. It follows, that if G is \mathbb{P} generic containing q then $\dot{U}^G \cap V[G_\alpha]$ is determined by G_α and therefore it is a normal ultrafilter U_α on κ in $V[G_\alpha]$. Now extend q once more to length $\alpha + 1$ by choosing \dot{U}_α to be the name for $U_\alpha = \dot{U}^G \cap V[G_\alpha]$.

This argument gives us the desired property for a single $\alpha < \Gamma$. To have it for all $\alpha \in S \cup \{\sup S\}$ for an S of order type κ^* , we just have to iterate the process κ^* -many times (this is possible because $\kappa^* \leq \Gamma$), and then by cofinality considerations we see that moreover $\dot{U} \cap V[G_{\sup S}] \in V[G_{\sup S}]$. \square

Remark: Note that we can choose the domains of our conditions such that they have size Γ .

Working in the Laver-prepared model $V[G_{S_\kappa}]$, take S to be a set with the properties of the lemma above; this set will be fixed for the rest of the paper.

Now, using our Laver preparation S_κ and Laver function h we choose a supercompactness embedding $j : V \rightarrow M$ with critical point κ satisfying $j(\kappa) \geq \lambda$ where $\lambda \geq |S_\kappa * \mathbb{P}|$, $M^\kappa \subseteq M$ and $j(h)(\kappa) = (\mathbb{P}, \lambda)$. Then $j(S_\kappa) = S_\kappa * \dot{\mathbb{P}} * \dot{S}^*$ for an appropriate tail iteration \dot{S}^* in M . Also if we denote $\mathbb{P}' = j(\mathbb{P})$ applying j to $S_\kappa * \dot{\mathbb{P}}$ we get $j(S_\kappa * \dot{\mathbb{P}}) = S_\kappa * \mathbb{P} * \dot{S}^* * (\mathbb{P}')^M$.

Consider then $j_0 : V[G_{S_\kappa}] \rightarrow M[G_{S_\kappa}][G_\mathbb{P}][H]$ where $G_{S_\kappa} * G_\mathbb{P} * H$ is generic for $j(S_\kappa * \dot{\mathbb{P}})$. We want to lift again to $j^* : V[G_{S_\kappa}][G_\mathbb{P}] \rightarrow M[G_{S_\kappa}][G_\mathbb{P}][H][G_{\mathbb{P}'}]$ where $\mathbb{P}' = j_0(\mathbb{P})$. We will do this by listing the maximal antichains below some master condition in \mathbb{P}' extending every condition of the form $j_0(p)$ for $p \in G_\mathbb{P}$. The obvious master condition comes from choosing a lower bound p_0^* of $j_0''G_\mathbb{P}$.²

This condition has support contained in $j''\Gamma^+$ and for each $i < \Gamma^+$ even chooses the filter name $\dot{U}_{j(i)}$ to be $j_0(\dot{U}_i)$ as well as a $j(\kappa)$ -Mathias condition with first component \check{x}_i , the Mathias generic added by $G_\mathbb{P}$ at stage i of

²This exists because $j_0''G_\mathbb{P}$ is directed and the forcing is sufficiently directed-closed.

the iteration. However we will choose a stronger master condition p^* with support contained in $j''\Gamma^+$ as follows.

(*) If $i < \Gamma^+$ is an even ordinal and for each $A \in \mathcal{U}_i$ there is a $G_{\mathbb{P}_i}$ -name \dot{X} such that $A = X^{G_{\mathbb{P}_i}}$ and a condition $p \in G_{\mathbb{P}_i}$ such that $j_0(p) \Vdash \kappa \in j_0(\dot{X})$, then $p^*(j(i))$ is obtained from $p_0^*(j(i))$ by replacing the first component x_i of its $j(\kappa)$ -Mathias condition by $x_i \cup \{\kappa\}$.

Otherwise $p^*(j(i)) = p_0^*(j(i))$.

Lemma 11. *The condition p^* is an extension of p_0^* . If $G_{\mathbb{P}}$ is chosen to contain p^* , j^* is the resulting lifting of j_0 and \mathcal{U} is the resulting normal ultrafilter on κ derived from j^* , then whenever \mathcal{U}_i is contained in \mathcal{U} , we have that $x_i \in \mathcal{U}$.*

Proof. To show the first claim, it is enough to show that for all $i < \Gamma^+$ the condition p_i^* defined as p^* but replacing $x_{j(l)}$ by $x_{j(l)} \cup \{\kappa\}$ for $l < i$ satisfying (*) extends p_0^* . We do this by induction on i . The base and limit cases are immediate. For the successor case, suppose we have the result for i and we want to prove it for $i+1$. Let $G_{\mathbb{P}_j^*}(i)$ be any generic containing $p_i^* \restriction j(i)$ and extend it to a generic $G_{\mathbb{P}^*}$ containing p_i^* . Hence, using the induction hypothesis $G_{\mathbb{P}^*}$ also contains p_0^* and therefore gives us a lifting j^* of j_0 .

Now, any $p \in G_{\mathbb{P}}$ can be extended (inside $G_{\mathbb{P}}$) so that the Mathias condition it specifies at stage i is of the form $(s, A) \in \mathbb{M}_{\mathcal{U}_i}^\kappa$ where $s \subseteq x_i$ and $A \in \mathcal{U}_i$. Then using (*) we infer $A = X^{G_{\mathbb{P}_i}}$ where $j_0(q) \Vdash \check{\kappa} \in j_0(\dot{X})$ for some $q \in G_{\mathbb{P}_i}$.

But then, since $p_0^* \in G_{\mathbb{P}^*}$, $j_0(q)$ is an element of $G_{\mathbb{P}_j^*}(i)$ and therefore

$$\kappa \in j_0(\dot{X})^{G_{\mathbb{P}_j^*}(i)} = j^*(A).$$

It follows that the $j(\kappa)$ -Mathias condition specified by $p_{i+1}^*(j(i))^{G_{\mathbb{P}^*}(j(i))}$ with first component $x_i \cup \{\kappa\}$ does extend

$$(x_i, j^*(A)) = (x_i, j_0(\dot{X})^{G_{\mathbb{P}_j^*}(i)}) \leq (s, j_0(\dot{X})^{G_{\mathbb{P}_j^*}(i)}).$$

This means that $p_i^* \restriction j(i) \Vdash p_{i+1}^*(j(i)) \leq (s, j_0(\dot{X})) = j_0(p)(j(i))$ and thus p_{i+1}^* extends $j_0^*(p)$ for each $p \in G_{\mathbb{P}}$ and then also extends p_0^* .

To see the second claim, note that if $\mathcal{U}_i \subseteq \mathcal{U}$, then $\kappa \in j^*(A)$ for all $A \in \mathcal{U}_i$ which implies that (*) is satisfied at i . Then $\kappa \in j^*(x_i)$ and so $x_i \in \mathcal{U}$. \square

Theorem 12. *Suppose κ is a supercompact cardinal and κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$, $\Gamma^\kappa = \Gamma$. There is a forcing notion \mathbb{P}^* preserving cofinalities such that $V^{\mathbb{P}^*} \models \mathfrak{u}(\kappa) = \kappa^* \wedge 2^\kappa = \Gamma$.*

Proof. We will not work with the whole generic extension given by \mathbb{P} . In fact we will chop the iteration at the step $\alpha = \sup(S)$ (as in the Lemma 10) which is an ordinal of cofinality κ^* . Define $\mathbb{P}^* = \mathbb{P}_\alpha$.

Take G to be a \mathbb{P}^* -generic filter. The fact that $2^\kappa = \Gamma$ in the extension is a consequence of the fact that the domains of the conditions obtained in Lemma 10 can be chosen in such a way that they all have size Γ .

To prove $\mathfrak{u}(\kappa) = \kappa^*$ we consider the ultrafilter \mathcal{U}^* on κ given by the restriction of \mathcal{U} (Lemma 10). Then by the same lemma note that for all $i \in S$ the restriction of \mathcal{U} to the model $V[G_i]$ belongs to $V[G_{i+1}]$ and moreover, this is the ultrafilter U_i^G chosen generically at stage i .

Furthermore by our choice of Master Conditions the κ -Mathias generics \dot{x}_i belong to \mathcal{U} . Then \mathcal{U}^* is generated by \dot{x}_i for $i \in S$.

The other inequality $\mathfrak{u}(\kappa) \geq \kappa^*$ is a consequence of $\mathfrak{b}(\kappa) \geq \kappa^*$ and Proposition 13. \square

Proposition 13. $\mathfrak{b}(\kappa) \leq \mathfrak{r}(\kappa)$ and $\mathfrak{r}(\kappa) \leq \mathfrak{u}(\kappa)$.

Proof. The first is the consequence of the following property that can be directly generalized from the countable case: there are functions $\Phi : [\kappa]^\kappa \rightarrow \kappa^{\uparrow\kappa}$ and $\Psi : \kappa^{\uparrow\kappa} \rightarrow [\kappa]^\kappa$ such that whenever $\Phi(A) \leq^* f$ then $\Psi(f)$ splits A .

For the second inequality, it is just necessary to notice that if \mathcal{B} is a base for a uniform ultrafilter on κ , then \mathcal{B} cannot be split by a single set X . Otherwise neither X nor $\kappa \setminus X$ will belong to the ultrafilter. \square

4. The generalized cardinal characteristics

In the following subsections 4.1 - 4.6 we systemize those properties of the generalized cardinal characteristics which will be of importance for our main consistency result.

4.1. Unbounded and Dominating Families in ${}^\kappa\kappa$

Definition 14. For two functions $f, g \in \kappa^\kappa$, we say $f \leq^* g$ if and only if there exists $\alpha < \kappa$ such that for all $\beta > \alpha$, $f(\beta) \leq g(\beta)$. A family \mathfrak{F} of functions from κ to κ is said to be

- *dominating*, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $g \leq^* f$.
- *unbounded*, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $f \not\leq^* g$.

Definition 15. The unbounding and dominating numbers, $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ respectively, are defined as follows:

- $\mathfrak{b}(\kappa) = \min\{|\mathfrak{F}|: \mathfrak{F} \text{ is an unbounded family of functions from } \kappa \text{ to } \kappa\}.$
- $\mathfrak{d}(\kappa) = \min\{|\mathfrak{F}|: \mathfrak{F} \text{ is a dominating family of functions from } \kappa \text{ to } \kappa\}.$

Definition 16 (Generalized Laver forcing). *Let \mathcal{U} be a κ -complete non-principal ultrafilter on κ .*

- *A \mathcal{U} -Laver tree is a κ -closed tree $T \subseteq \kappa^{<\kappa}$ of increasing sequences with the property that $\forall s \in T (|s| \geq |\text{stem}(T)| \rightarrow \text{Succ}_T(s) \in \mathcal{U})$.*
- *The generalized Laver Forcing $\mathbb{L}_{\mathcal{U}}^{\kappa}$ consists of all \mathcal{U} -Laver trees with order given by inclusion.*

Proposition 17. *Generalized Laver forcing $\mathbb{L}_{\mathcal{U}}^{\kappa}$ generically adds a dominating function from κ to κ .*

Proof. Let G be a $\mathbb{L}_{\mathcal{U}}^{\kappa}$ -generic filter. The Laver generic function in κ^{κ} , l_G , is defined as follows: $l_G = \cap\{[T] : T \in G\}$ where $[T]$ is the set of branches in T .

To show that l_G is a dominating function it is enough to notice that, for all $f \in \kappa^{\kappa}$ and all $T \in \mathbb{L}_{\mathcal{U}}^{\kappa}$, the set $T_f = \{s \in T : \forall \alpha (|\text{stem}(T)| \leq \alpha < |s| \rightarrow s(\alpha) > f(\alpha))\}$ is also a condition in $\mathbb{L}_{\mathcal{U}}^{\kappa}$ and $T_f \leq T$. By genericity we conclude that $V[G] \models \forall f \in V \cap \kappa^{\kappa} (f \leq^* l_G)$. \square

Lemma 18. *If \mathcal{U} is a normal ultrafilter on κ , then $\mathbb{M}_{\mathcal{U}}^{\kappa}$ and $\mathbb{L}_{\mathcal{U}}^{\kappa}$ are forcing equivalent.*

Proof. The main point that we will use in this proof is that, when \mathcal{U} is normal we have the following ‘‘Ramsey’’-like property: for all $f : [\kappa]^{<\omega} \rightarrow \gamma$ where $\gamma < \kappa$, there is a set in \mathcal{U} homogeneous for f (see for example [11, Theorem 7.17]). Also it is worth remembering that in the countable case if \mathcal{U} is a Ramsey Ultrafilter $\mathbb{M}(\mathcal{U}) \simeq \mathbb{L}(\mathcal{U})$.

We want to define a dense embedding $\varphi : \mathbb{M}_{\mathcal{U}}^{\kappa} \rightarrow \mathbb{L}_{\mathcal{U}}^{\kappa}$. Take (s, A) a condition in $\mathbb{M}_{\mathcal{U}}^{\kappa}$ and define the tree $T = T_{(s,A)}$ as follows:

- $\sigma = \text{stem}(T)$ will be the increasing enumeration of s .
- If we already have constructed $\tau \in T_{\alpha}$, with $\tau \supseteq \sigma$, then $\tau \hat{\smallfrown} \langle \gamma \rangle \in T_{\alpha+1}$ if and only if $\gamma \in A$ and $\gamma \geq \sup\{\tau(\beta) + 1 : \beta < \gamma\}$.
- In the limit steps just ensure that $\tau \in T_{\alpha}$ if and only if $\tau \restriction \beta \in T_{\beta}$ for all $\beta < \alpha$.

Note that T is a condition in $\mathbb{L}_{\mathcal{U}}^\kappa$. For the limit steps note that if $\tau \in T_\alpha$ for α limit, then the set $\text{Succ}_T(\tau) = \bigcap_{\beta < \alpha} \text{Succ}_T(\tau \restriction \beta)$.

Now, consider the map $\varphi : (s, A) \rightarrow T_{(s, A)}$. Since this map preserves \leq , it is enough to prove that the trees of the form $T_{(s, A)}$ are dense in $\mathbb{L}_{\mathcal{U}}^\kappa$. For that, take an arbitrary $T \in \mathbb{L}_{\mathcal{U}}^\kappa$ and define for $\alpha \leq \beta$:

$$f(\{\alpha, \beta\}) = \begin{cases} 1 & \text{if } \forall s \in T \text{ with } \alpha \geq \sup\{s(\gamma) : \gamma < |s|\} \\ & (\beta \in \text{Succ}_T(s)) \\ 0 & \text{otherwise} \end{cases}$$

Using the Ramsey-like property we can find a set $B \in \mathcal{U}$ homogeneous for f . The color of B cannot be 0 because T is a Laver tree. Now, knowing that $f''[B]^2 = \{1\}$, we can define $s = \text{ran}(\text{stem}(T))$ and $A = B \cap \text{Succ}_T(\text{stem}(T))$ and conclude that $T_{(s, A)} \leq T$ as we wanted. \square

Corollary 19. *If \mathcal{U} is a normal ultrafilter on κ then $\mathbb{M}_{\mathcal{U}}^\kappa$ always adds dominating functions.*

4.2. κ -maximal almost disjoint families

Definition 20. *Two sets A and $B \in \mathcal{P}(\kappa)$ are called κ -almost disjoint if $A \cap B$ has size $< \kappa$. We say that a family of sets $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ is κ -almost disjoint if it has size at least κ and all its elements are pairwise κ -almost disjoint. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called κ -maximal almost disjoint family (abbreviated κ -mad) if it is κ -almost disjoint and is not properly included in another such family.*

Definition 21. $\mathfrak{a}(\kappa) = \min\{|\mathfrak{A}| : \mathfrak{A} \text{ is a } \kappa\text{-mad family}\}$

Proposition 22. $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$

Proof. Let $\mathfrak{A} = \{A_\alpha : \alpha < \lambda\}$ be a κ -almost disjoint family where $\lambda < \mathfrak{b}(\kappa)$. For each $\alpha < \kappa$, let $\tilde{A}_\alpha = A_\alpha \setminus \bigcup_{\delta < \alpha} (A_\alpha \cap A_\delta)$. Since \mathfrak{A} is κ -ad, we have $|\tilde{A}_\alpha| = \kappa$, also $\tilde{A}_\alpha \cap \tilde{A}_\beta = \emptyset$ for all $\alpha, \beta < \kappa$. Thus, $\tilde{A}_\alpha =^* A_\alpha$. (Here $*$ means modulo a set of size $< \kappa$).

Whenever $g \in \kappa^\kappa$, define $e_g^\alpha = \text{next}(\tilde{A}_\alpha, g(\alpha))$, the least ordinal in \tilde{A}_α greater than $g(\alpha)$. Let $E_g = \{e_g^\alpha : \alpha < \kappa\}$. Then E_g contains one element of each \tilde{A}_α , so it is unbounded in κ . Also $|E_g \cap A_\alpha| < \kappa$, for all $\alpha < \kappa$.

Now consider those α such that $\kappa \leq \alpha < \lambda$. Each $A_\alpha \cap A_\gamma$ has size less than κ , so we can fix f_α such that for all $\gamma < \kappa$ all elements of $A_\alpha \cap A_\gamma$ are less than $f_\alpha(\gamma)$: take $f_\alpha(\gamma) = \sup(A_\alpha \cap A_\gamma) + 1$.

Now consider $\{f_\alpha : \kappa \leq \alpha < \lambda\}$, which is a family of $\lambda < \mathfrak{b}(\kappa)$ functions. As a consequence there exists $g \in \kappa^\kappa$ with the property $f_\alpha <^* g$, for all $\alpha \geq \kappa$.

Thus we have that $E_g \cap A_\alpha$ has size less than κ , for all α because if $e_g^\gamma \in E_g \cap A_\alpha$ then $e_g^\gamma \in A_\gamma$ and $e_g^\gamma > g(\gamma)$, so $f_\alpha(\gamma) > e_g^\gamma > g(\gamma)$ which is only possible for γ in a set of less than κ values.

Therefore, \mathfrak{A} is not maximal. So, we have $\mathfrak{b}(\kappa) \leq \lambda$. \square

Definition 23. Let $\mathcal{A} = \{A_i\}_{i < \delta}$ be a κ -almost disjoint family. Let $\bar{\mathbb{Q}}(\mathcal{A}, \kappa)$ be the poset of all pairs (s, F) where $s \in 2^{<\kappa}$ and $F \in [\mathcal{A}]^{<\kappa}$, with extension relation stating that $(t, H) \leq (s, F)$ if and only if $t \supseteq s$, $H \supseteq F$ and for all $i \in \text{dom}(t) \setminus \text{dom}(s)$ with $t(i) = 1$ we have $i \notin \bigcup\{A : A \in F\}$

Note that the poset $\bar{\mathbb{Q}}(\mathcal{A}, \kappa)$ is κ -centered and κ -directed closed. If G is $\bar{\mathbb{Q}}(\mathcal{A}, \kappa)$ -generic then $\chi_G = \bigcup\{t : \exists F(t, F) \in G\}$ is the characteristic function of an unbounded subset x_G of κ such that $\forall A \in \mathcal{A} (|A \cap x_G| < \kappa)$.

Proposition 24. If $Y \in [\kappa]^\kappa \setminus \mathcal{I}_\mathcal{A}$, where $\mathcal{I}_\mathcal{A}$ is the κ -complete ideal generated by the κ -ad-family \mathcal{A} , then $\Vdash_{\bar{\mathbb{Q}}(\mathcal{A}, \kappa)} |Y \cap \dot{x}_G| = \kappa$.

Proof. Let $(s, F) \in \bar{\mathbb{Q}}(\mathcal{A}, \kappa)$ and $\alpha < \kappa$ be arbitrary. It is sufficient to show that there are $(t, H) \leq (s, F)$ and $\beta > \alpha$ such that $(t, H) \Vdash \beta \in \dot{Y} \cap \dot{x}_G$. Since $Y \notin \mathcal{I}_\mathcal{A}$, we have that $|Y \setminus \bigcup F| = \kappa$. Take any $\beta > \alpha$ in $Y \setminus \bigcup F$ and define $t' = t \cup \{(\beta, 1)\} \cup \{(\gamma, 0) : \sup(\text{dom}(t)) < \gamma < \beta\}$. Then (t', H) is as desired. \square

4.3. The Generalized Splitting, Reaping and Independence Numbers

Definition 25. For A and $B \in \mathcal{P}(\kappa)$, say $A \subseteq^* B$ (A is almost contained in B) if $A \setminus B$ has size $< \kappa$. We also say that A splits B if both $A \cap B$ and $B \setminus A$ have size κ . A family \mathcal{A} is called a splitting family if every unbounded (with supremum κ) subset of κ is split by a member of \mathcal{A} . Finally \mathcal{A} is unsplit if no single set splits all members of \mathcal{A} .

- $\mathfrak{s}(\kappa) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a splitting family of subsets of } \kappa\}$.
- $\mathfrak{r}(\kappa) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is an unsplit family of subsets of } \kappa\}$.

Definition 26. A family $\mathcal{I} = \{I_\delta : \delta < \mu\}$ of subsets of κ is called κ -independent if for all disjoint $I_0, I_1 \subseteq \mathcal{I}$, both of size $< \kappa$, $\bigcap_{\delta \in I_0} I_\delta \cap \bigcap_{\delta \in I_1} (I_\delta)^c$ is unbounded in κ . The generalized independence number $\mathfrak{i}(\kappa)$ is defined as the minimal size of a κ -independent family.

Even though we do not know the exact value of $\mathfrak{i}(\kappa)$ in the model we construct (see Theorem 35), the inequality which we prove below and the value we fix for $\mathfrak{d}(\kappa)$ in this model, will provide a constraint for $\mathfrak{i}(\kappa)$.

Proposition 27. *If $\mathfrak{d}(\kappa)$ is such that for every $\gamma < \mathfrak{d}(\kappa)$ we have $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$, then $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$*

The proof will be essentially a modification of the one for the countable case (Theorem 5.3 in [1]). To obtain the above proposition, we will need the following lemma.

Lemma 28. *Suppose $\mathcal{C} = (C_\alpha : \alpha < \kappa)$ is a \subseteq^* -decreasing sequence of unbounded subsets of κ and \mathcal{A} is a family of less than $\mathfrak{d}(\kappa)$ many subsets of κ such that each set in \mathcal{A} intersects every C_α in a set of size κ . Then \mathcal{C} has a pseudointersection B that also has unbounded intersection with each member of \mathcal{A} .*

Proof. Without loss of generality assume that the sequence \mathcal{C} is \subseteq -decreasing. For any $h \in \kappa^\kappa$ define $B_h = \bigcup_{\alpha < \kappa} (C_\alpha \cap h(\alpha))$, clearly B_h is a pseudointersection of \mathcal{C} . Thus, we must find $h \in \kappa^\kappa$ such that $|B_h \cap A| = \kappa$ for each $A \in \mathcal{A}$.

For each $A \in \mathcal{A}$ define the function $f_A \in \kappa^\kappa$ as follows: $f_A(\beta) =$ the β -th element of $A \cap C_\beta$. The set $\{f_A : A \in \mathcal{A}\}$ has cardinality $< \mathfrak{d}(\kappa)$, then we can find $h \in \kappa^\kappa$ such that for all $A \in \mathcal{A}$, $h \not\leq^* f_A$ (i.e. $X_A = \{\delta < \kappa : f_A(\delta) < h(\delta)\}$ is unbounded).

Then B_h will be the pseudointersection we need. Note that $B_h \cap A = \bigcup_{\alpha < \kappa} (C_\alpha \cap A) \cap h(\alpha) \supseteq \bigcup_{\alpha \in X_A} (C_\alpha \cap A) \cap f_A(\alpha)$ which is unbounded. \square

Proof of Proposition 27. Suppose that \mathcal{I} is an independent family of cardinality $< \mathfrak{d}(\kappa)$, we will show it is not maximal. For this purpose choose $\mathcal{D} = (D_\alpha : \alpha < \kappa) \subseteq \mathcal{I}$ and let $\mathcal{I}' = \mathcal{I} \setminus \mathcal{D}$.

For each $f : \kappa \rightarrow 2$ consider the set $C_\alpha = \bigcap_{\beta < \alpha} D_\beta^{f(\beta)}$ where $D^0 = D$ and $D^1 = D^c$, also define $\mathcal{A} = \{\bigcap I_0 \setminus \bigcup I_1 : I_0 \text{ and } I_1 \text{ are disjoint subfamilies of } \mathcal{I} \text{ of size } < \kappa\}$. Since $|\mathcal{I}|^{<\kappa} < \mathfrak{d}(\kappa)$, the family \mathcal{A} has size $< \mathfrak{d}(\kappa)$.

Then, using the lemma before there exists a pseudointersection B_f of the family $(C_\alpha : \alpha < \kappa)$ that intersects in an unbounded set all members of \mathcal{A} . Then if $f \neq g$ we have $|B_f \cap B_g| < \kappa$ (Moreover, we can suppose they are disjoint).

Now, fix two disjoint dense subsets X and X' of 2^κ . Take $Y = \bigcup_{f \in X} B_f$ and $Y' = \bigcup_{f \in X'} B_f$, note that $Y \cap Y' = \emptyset$. Then it is enough to show that both Y and Y' have intersection of size κ with each member of \mathcal{A} . We write the argument for Y (for Y' it is analogous).

Take $J_0, J_1 \subseteq \mathcal{I}$ both of size $< \kappa$, call J'_0, J'_1 their intersections with \mathcal{I}' . There exists $\alpha < \kappa$ such that if D_β belongs to J_0 or J_1 , then $\beta < \alpha$ and using the density of the sets X fix $f \in X$ such that, if $D_\beta \in J_0 \cup J_1$, then $f(\beta) = 0$ or 1 respectively. Hence:

$$\begin{aligned} \bigcap J_0 \setminus \bigcup J_1 &= \bigcap J'_0 \setminus \bigcup J'_1 \cap \bigcap_{\{\beta: D_\beta \in J_0 \cup J_1\}} D_\beta^{f(\beta)} \\ &\supseteq \bigcap J'_0 \setminus \bigcup J'_1 \cap \bigcap_{\beta < \alpha} D_\beta^{f(\beta)} \\ &\stackrel{*}{\supseteq} \bigcap J'_0 \setminus \bigcup J'_1 \cap B_f \text{ which is unbounded.} \quad (1) \end{aligned}$$

□

4.4. The generalized pseudointersection and tower numbers

Definition 29. Let \mathcal{F} be a family of subsets of κ , we say that \mathcal{F} has the strong intersection property (SIP) if any subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size $< \kappa$ has intersection of size κ , we also say that $A \subseteq \kappa$ is a pseudointersection of \mathcal{F} is $A \subseteq^* F$, for all $F \in \mathcal{F}$. A tower \mathcal{T} is a well-ordered family of subsets of κ with the SIP that has no pseudointersection of size κ .

- The generalized pseudointersection number $\mathfrak{p}(\kappa)$ is defined as the minimal size of a family \mathcal{F} which has the SIP but no pseudointersection of size κ .
- The generalized tower number $\mathfrak{t}(\kappa)$ is defined as the minimal size of a tower \mathcal{T} of subsets of κ .

Lemma 30. $\kappa^+ \leq \mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$

Proof. First we prove $\kappa^+ \leq \mathfrak{p}(\kappa)$: Take a family of subsets of κ , $\mathcal{B} = (B_\alpha : \alpha < \kappa)$ with the SIP. Then we can construct a new family $\mathcal{B}' = (B'_\alpha : \alpha < \kappa)$ such that $B'_{\alpha+1} \subseteq B'_\alpha$ and $B'_\alpha \subseteq B_\alpha$ for all $\alpha < \kappa$. Simply define $B'_0 = B_0$, $B'_{\alpha+1} = B_{\alpha+1} \cap B'_\alpha$ and for limit γ , $B'_\gamma = \bigcap_{\alpha < \gamma} B'_\alpha$. Note that this construction is possible thanks to the SIP.

Then, without loss of generality we can find κ -many indexes β where it is possible to choose $a_\beta \in B'_\alpha \setminus B'_{\alpha+1}$. Hence the set $X = \{a_\beta : \beta < \kappa\}$ is a pseudointersection of the family \mathcal{B}' and so of \mathcal{B} .

$\mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa)$ is immediate from the definitions and $\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$ was proven in [9, Claim 1.8]. □

4.5. The generalized distributivity number

In order to generalize the distributivity number we have to consider a modification of it. In the countable case \mathfrak{h} is defined as the minimum cardinal λ such that the poset $\mathcal{P}(\omega)/\text{fin}$ is not λ -distributive, where a poset \mathbb{P} is λ -distributive if every collection \mathcal{D} of λ -many dense open sets has dense open intersection.

If we consider the poset $\mathcal{P}(\kappa)/<\kappa$, then it is not clear that the minimum cardinal λ such that it is not λ -distributive is greater even than ω . Hence we will start from a non-principal κ -complete ultrafilter \mathcal{W} , and consider it as a poset ordered by inclusion modulo the ideal of sets of size less than κ .

Definition 31. *The Generalized Distributivity Number with respect to the κ -complete ultrafilter \mathcal{W} , $\mathfrak{h}_{\mathcal{W}}(\kappa)$ is defined as the minimal $\lambda \geq \kappa$ for which \mathcal{W} is not λ -distributive.*

Proposition 32. $\mathfrak{p}(\kappa) \leq \mathfrak{h}_{\mathcal{W}}(\kappa) \leq \mathfrak{s}(\kappa)$.

Proof.

• $\mathfrak{p}(\kappa) \leq \mathfrak{h}_{\mathcal{W}}(\kappa)$: Let $\lambda < \mathfrak{p}(\kappa)$. We shall prove that \mathcal{F} is λ -distributive. For this purpose take $(D_\alpha : \alpha < \lambda)$ to be a family of open dense sets in \mathcal{W} .

Using density as well as the completeness of the filter \mathcal{W} it is possible to construct a sequence $\bar{X} = (X_\alpha : \alpha < \lambda)$ such that, for all $\alpha < \lambda$, $X_\alpha \in D_\alpha$. By induction start with $X_0 \in D_0$ and given X_α , take $X_{\alpha+1} \in D_{\alpha+1}$ where $X_{\alpha+1} \leq X_\alpha$. For the limit step $\alpha < \lambda$, take $Y = \bigcap_{\beta < \alpha} X_\beta \in \mathcal{W}$ and then $X_\alpha \in D_\alpha$, $X_\alpha \leq Y$.

Thus the family \bar{X} has the SIP and since $\lambda < \mathfrak{p}(\kappa)$ has a pseudointersection $X \in \mathcal{W}$ which belongs to $\bigcap_{\alpha < \lambda} D_\alpha$.

• $\mathfrak{h}_{\mathcal{F}}(\kappa) \leq \mathfrak{s}(\kappa)$: Let \mathcal{S} be a splitting family of subsets of κ . For each $S \in \mathcal{S}$, the set $D_S = \{X \in \mathcal{W} : X \text{ is not split by } S\}$ is dense open. Because \mathcal{S} is a splitting family we obtain $\bigcap_{S \in \mathcal{S}} D_S = \emptyset$. \square

4.6. Cardinals from Cichón's diagram at κ

Assume κ is uncountable and satisfies $\kappa^{<\kappa} = \kappa$. It is possible to endow 2^κ with the topology generated by the sets of the form $[s] = \{f \in 2^\kappa : f \supseteq s\}$, for $s \in 2^{<\kappa}$. Then it is possible to define nowhere dense sets and meager sets as κ -unions of nowhere dense sets. Hence, we can consider the Meager Ideal \mathcal{M}_κ and study the cardinal characteristics associated to this ideal. Specifically we are interested in the cardinals in Cichón's Diagram.

- $\text{add}(\mathcal{M}_\kappa) = \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{M}_\kappa \text{ and } \bigcup \mathcal{J} \notin \mathcal{M}_\kappa\}$

- $\text{cov}(\mathcal{M}_\kappa) = \min\{|\mathcal{J}|: \mathcal{J} \subseteq \mathcal{M}_\kappa \text{ and } \cup \mathcal{J} = 2^\kappa\}$
- $\text{cof}(\mathcal{M}_\kappa) = \min\{|\mathcal{J}|: \mathcal{J} \subseteq \mathcal{M}_\kappa \text{ and } \forall M \in \mathcal{M}_\kappa \exists J \in \mathcal{J} \text{ s.t. } M \subseteq J\}$
- $\text{non}(\mathcal{M}_\kappa) = \min\{|X|: X \subset 2^\kappa \text{ and } X \notin \mathcal{M}_\kappa\}$

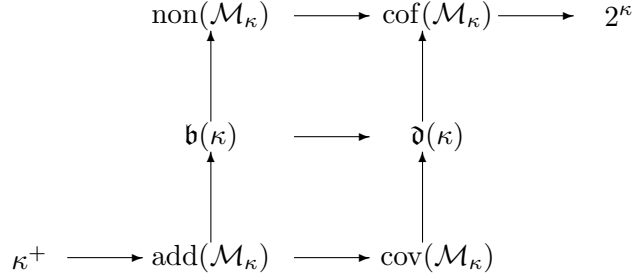


Figure 1: Generalization of Cichón's diagram (for κ strongly inaccessible)

If in addition κ is strongly inaccessible we have a similar diagram as in the countable case (for specific details about these properties see [3]). Also, the following well known relationships between the classical cardinal characteristics (see for example [1]) hold.

Lemma 33.

$\text{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}(\kappa), \text{cov}(\mathcal{M}_\kappa)\}$ and $\text{cof}(\mathcal{M}_\kappa) = \max\{\mathfrak{d}(\kappa), \text{non}(\mathcal{M}_\kappa)\}$.

5. Applications

Until the end of the paper let κ , κ^* , Γ , α and \mathbb{P}^* be fixed as in Theorem 12.

Theorem 34. *Let G be \mathbb{P}^* -generic. Then $V[G]$ satisfies $\text{add}(\mathcal{M}_\kappa) = \text{cof}(\mathcal{M}_\kappa) = \text{non}(\mathcal{M}_\kappa) = \text{cov}(\mathcal{M}_\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{b}(\kappa) = \kappa^*$.*

Proof. Note that $\mathfrak{b}(\kappa) \geq \kappa^*$ because any set of functions in κ^κ of size $< \kappa^*$ appears in some initial part of the iteration (by Lemma 8) and so is dominated by the Mathias generic functions added at later stages. On the other hand, any cofinal sequence of length κ^* of the Mathias generics forms a dominating family. Thus $\mathfrak{d}(\kappa) \leq \kappa^*$ and since clearly $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$, we obtain $V^{\mathbb{P}^*} \models \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \kappa^*$.

To see that $\mathfrak{s}(\kappa) \geq \kappa^*$, observe that a Mathias generic subset of κ is unsplit by any ground model subset of κ and that every family of κ -reals of

size less than κ^* is contained in $V^{\mathbb{P}_\beta}$ for some $\beta < \alpha$. On the other hand, since \mathbb{P}^* is locally a $(< \kappa)$ -support iteration, it does add κ -Cohen reals. Any cofinal sequence of length κ^* of such κ -Cohen reals forms a splitting family and so $V^{\mathbb{P}^*} \models \mathfrak{s}(\kappa) \leq \kappa^*$. Thus $V^{\mathbb{P}^*} \models \mathfrak{s}(\kappa) = \kappa^*$. That $\mathfrak{r}(\kappa) = \kappa^*$ follows from Proposition 13.

To verify the values of the characteristics associated to \mathcal{M}_κ , proceed as follows. Since $\mathfrak{b}(\kappa) \leq \text{non}(\mathcal{M}_\kappa)$, $V^{\mathbb{P}^*} \models \kappa^* \leq \text{non}(\mathcal{M}_\kappa)$. On the other hand, any cofinal sequence of κ -Cohen reals of length κ^* is a non-meager set and so a witness to $\text{non}(\mathcal{M}_\kappa) \leq \kappa^*$. By a similar argument and the fact that $\mathfrak{d}(\kappa) = \kappa^*$ in $V^{\mathbb{P}^*}$, we obtain that $V^{\mathbb{P}^*} \models \text{cov}(\mathcal{M}_\kappa) = \kappa^*$. Now, Lemma 33 implies that $\text{add}(\mathcal{M}_\kappa) = \kappa^* = \text{cof}(\mathcal{M}_\kappa)$. \square

Now, we are ready to prove our main theorem.

Theorem 35. *Suppose κ is a supercompact cardinal, κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$ and Γ satisfies $\Gamma^\kappa = \Gamma$. Then there is forcing extension in which cardinals have not been changed satisfying:*

$$\begin{aligned} \kappa^* &= \mathfrak{u}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \text{cov}(\mathcal{M}_\kappa) \\ &= \text{add}(\mathcal{M}_\kappa) = \text{non}(\mathcal{M}_\kappa) = \text{cof}(\mathcal{M}_\kappa) \text{ and } 2^\kappa = \Gamma. \end{aligned}$$

If in addition $(\Gamma)^{<\kappa^*} = \Gamma$ then we can also provide that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{h}_\mathcal{W}(\kappa) = \kappa^*$ where \mathcal{W} is a κ -complete ultrafilter on κ .

Proof. We will modify the iteration \mathbb{P}^* to an iteration $\bar{\mathbb{P}}^*$ by specifying the iterands \mathbb{Q}_j for every odd ordinal $j < \alpha$. Let $\bar{\gamma} = \langle \gamma_i \rangle_{i < \kappa^*}$ be a strictly increasing cofinal in α sequence of odd ordinals. The stages in $\bar{\gamma}$ will be used to add a κ -maximal almost disjoint family of size κ^* .

If $\Gamma^{<\kappa^*} = \Gamma$, then using an appropriate bookkeeping function F with domain the odd ordinals in α which are not in the cofinal sequence $\bar{\gamma}$ we can use the generalized Mathias poset to add pseudointersections to all filter bases of size $< \kappa^*$ with the SIP. In case $\Gamma^{<\kappa^*} > \Gamma$, just take for odd stages which are not in $\bar{\gamma}$ arbitrary κ -centered, κ -directed closed forcing notions of size at most Γ , such as the trivial forcing.

To complete the definition of $\bar{\mathbb{P}}^*$ it remains to specify the stages in $\bar{\gamma}$. Fix a ground model κ -ad family \mathcal{A}_0 of size κ and let $\mathbb{Q}_{\gamma_0} = \bar{\mathbb{Q}}(\mathcal{A}_0, \kappa)$ (see Definition 23). Now, fix any $i < \kappa^*$. For each $j < i$ let \bar{x}_{γ_j} be the generic subset of κ added by $\mathbb{Q}_{\gamma_j} = \bar{\mathbb{Q}}(\mathcal{A}_j, \kappa)$ where $\mathcal{A}_j = \mathcal{A}_0 \cup \{\bar{x}_{\gamma_k}\}_{k < j}$.

With this the recursive definition of the iteration $\bar{\mathbb{P}}^*$ is defined. In $V^{\bar{\mathbb{P}}^*}$ let $\mathcal{A}_* = \mathcal{A}_0 \cup \{\bar{x}_{\gamma_j}\}_{j < \kappa^*}$. We will show that \mathcal{A}_* is a κ -mad family. Clearly \mathcal{A}_* is κ -ad. To show maximality of \mathcal{A}_* , consider an arbitrary $\bar{\mathbb{P}}^*$ -name \dot{X} for

a subset of κ and suppose $\Vdash_{\mathbb{P}^*} (\{\dot{X}\} \cup \mathcal{A}_* \text{ is } \kappa\text{-ad})$. By Corollary 8 \dot{X} can be viewed as a \mathbb{P}^*_β -name for some $\beta < \alpha$. Then for $\gamma_j > \beta$, by Proposition 24 we obtain $V^{\mathbb{P}^*_\beta} \models |\bar{x}_{\gamma_j} \cap \dot{X}| = \kappa$, which is a contradiction. Thus \mathcal{A}_* is indeed maximal and so $\mathfrak{a}(\kappa) \leq \kappa^*$. However in $V^{\mathbb{P}^*}$, $\mathfrak{b}(\kappa) = \kappa^*$ and since $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$ (Proposition 24) we obtain $V^{\mathbb{P}^*} \models \mathfrak{a}(\kappa) = \kappa^*$.

Suppose $\Gamma^{<\kappa^*} \leq \Gamma$. In this case, every filter of size $< \kappa^*$ with the SIP has a pseudointersection in $V^{\mathbb{P}^*}$. Thus in the final extension $\mathfrak{p}(\kappa) \geq \kappa^*$. However $\mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa) \leq \mathfrak{s}(\kappa)$ and since $V^{\mathbb{P}^*} \models \mathfrak{s}(\kappa) = \kappa^*$, we obtain that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \kappa^*$. By Proposition 32, $\mathfrak{h}(\kappa) \leq \mathfrak{s}(\kappa) = \kappa^*$ and $\kappa^* = \mathfrak{t}(\kappa) \leq \mathfrak{h}(\kappa)$. Thus $\mathfrak{h}(\kappa) = \kappa^*$. \square

The above iteration can be additionally modified so that in the final extension the minimal size of a κ -maximal cofinitary group, $\mathfrak{a}_g(\kappa)$, is κ^* . Indeed, one can use the stages in $\bar{\gamma}$ and [7, Definition 2.2.] to add a κ -maximal cofinitary group of size κ^* . The fact that $\kappa^* \leq \mathfrak{a}_g(\kappa)$ follows from $\mathfrak{b}(\kappa) \leq \mathfrak{a}_g(\kappa)$ (see [2]). Of interest remains the following question: can the above construction be modified to evaluate also $\mathfrak{i}(\kappa)$?

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